Poisson Processes (to model counting of event occurances as a function of time)

* first a more generalized counting process:

Def. a RP $N(t)$, $t \geq 0$, is a <u>counting process</u> iff

(i) $N(t) \geq 0 \quad \forall t \geq 0$

(ii) $N(t)$ is integer-valued

(iii) If $s < t$, then $N(s) \leq N(t)$

(iv) For $s < t$, $N(t) - N(s)$ is the # of event occurances in $(s, t]$

Def. a counting process $N(t)$, $t \geq 0$, is a <u>Poisson Process</u> with rate $\lambda > 0$ if:

(i) $N(0) = 0$

(ii) $N(t)$ has <u>independent increments</u>

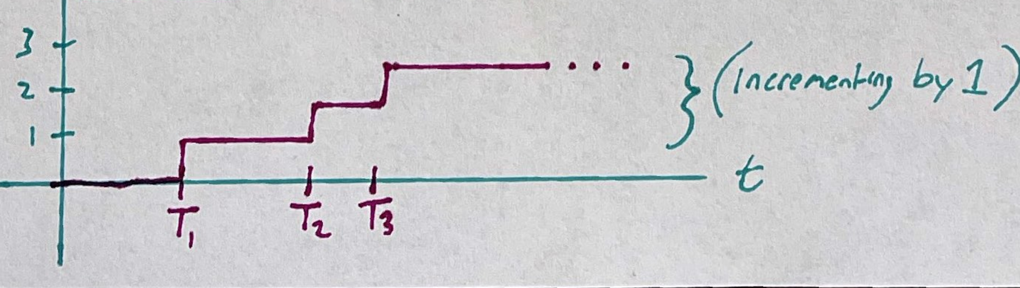
i.e., for any k & any $t_1 < t_2 < \ldots < t_k$, the RVs $N(t_2) - N(t_1)$, $N(t_3) - N(t_2)$, $\ldots$, $N(t_k) - N(t_{k-1})$ are independent

(iii) The # of occurances in any interval of length t is a Poisson RV with mean/weight of $\cancel{(1/\lambda)}$ (λt)

$$\Rightarrow \boxed{P(N(t+s) - N(s) = n) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}, \quad n = 0, 1, \ldots; \ \forall s, t \geq 0, \ s < t}$$

(* lambda is now the rate of occurances per unit of time)

$N(t) \sim$ Poisson Process



Interval Times of a Poisson Process:

$T_1, T_2, T_3, \ldots$ are called the "arrival times" of the process. However, the model for this sequence is somewhat complicated. Instead, the "interarrival times" are used.

* Let $\left| \begin{array}{l} X_1 = T_1 \quad \leftarrow \text{(RV } X_1 \text{ is simply the time of the first event)} \\ X_n = T_n - T_{n-1}, \; n = 2, 3, \ldots \end{array} \right.$

* The sequence X_n is the sequence of inter-arrival times
* It may be shown that the RVs $X_1, X_2, \ldots$ are i.i.d. exponential RVs with mean $(1/\lambda)$ ← (Unit of time per event occurance)

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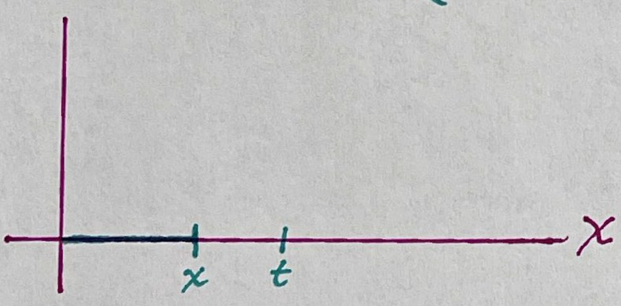
* Example: Let $N(t)$ be a Poisson Process, where 1 arrival/occurance occurs in the interval $[0, t]$ for some $t > 0$, let X be the time of this arrival in $[0, t]$

⇒ What is the distribution of X?

> * Given k arrivals occur in $[0, t]$, then $X_1, \ldots, X_k$ are independent & uniform on $[0, t]$

$$\text{For} \begin{cases} x \le 0, & P(X \le x) = 0 \\ x > t, & P(X \le x) = 1 \\ 0 \le x \le t, & P(X \le x) = P(N(x) = 1 \mid N(t) = 1) \end{cases}$$



$$= \frac{P(N(x) = 1, N(t) = 1)}{P(N(t) = 1)}$$

$$\text{So } P(X \le x) = \frac{P(N(x) = 1, N(t) - N(x) = 0)}{P(N(t) = 1)} = \frac{P(N(x) = 1)\, P(N(t) - N(x) = 0)}{P(N(t) = 1)}$$

$$= \frac{\lambda x \exp(-\lambda x) \exp(-\lambda(t - x))}{\lambda t \exp(-\lambda t)} = \frac{x}{t} \leftarrow \left(\text{CDF of a uniform distribution on } [0, t]\right)$$