Frequency Spectrum of a WSS RP

* Often useful to work w/ a RP in the frequency domain rather than the time domain
* This differs from the characteristic & moment generating functions

(Fourier Transform of the density function for a random variable)

"freq. domain-esque" but does not relate to frequency in terms of how a signal varies over time

(Similar to using the frequency domain for deterministic signals)

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* Convolution may be avoided when finding $R_Y(\tau) = (\tilde{h} * h * R_X)(\tau)$ ← (LTI System)
* Filter design for a deterministic signal is typically easiest when working with the frequency domain representation (e.g., to filter noise, design a Low-Pass Filter & set cutoff frequency)
* Noise in audio signals (e.g., white, pink, blue noise, etc.) ← (defined in the freq. domain for audio signals)

How is the frequency spectrum defined for a Random Process?

⇒ Instead of finding the FT for each sample realization of the RP, use expectation to represent the frequency spectrum as a whole

(depends on the autocorrelation function)

* Define the frequency spectrum in terms of the autocorrelation function of the process

Power Spectral Density (PSD)

Def.: The <u>PSD</u> of a WSS RP $X(t)$ is

$$\boxed{S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-i\omega\tau} \, d\tau \leftarrow (\text{FT of } R_X)} \quad \omega \in \mathbb{R}$$

(frequency variable, not sample space)

* Since $R_X(-\tau) = R_X(\tau)$, ("R_X is an even function in τ") $S_X(\omega)$ is <u>real-valued</u>
* $R_X(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) e^{i\omega\tau} \, d\omega \leftarrow$ (<u>Inverse FT</u>)
* Since $R_X(\tau)$ is real, $S_X(\omega)$ is an even function of ω, $S_X(-\omega) = S_X(\omega)$, $\forall \omega \in \mathbb{R}$

Def.: The Cross Power Spectral Density of jointly WSS RP $X(t)$ & $Y(t)$ is $\boxed{S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau} \, d\tau}$

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* Recall that if $X(t) \rightarrow \boxed{h(t)} \rightarrow Y(t)$ then $R_Y(\tau) = (\tilde{h} * h * R_X)(\tau)$
(WSS) (WSS)

* Let the <u>frequency response</u> be the FT of the impulse response: $H(\omega) = \mathcal{F}\{h(t)\}$

$$\Rightarrow \mathcal{F}\{\tilde{h}(t)\} = \mathcal{F}\{h(-t)\} = H^*(\omega) \leftarrow (\text{Complex conjugate})$$

* Even though $S_X(\omega)$ & $S_Y(\omega)$ are real-valued, <u>$H(t)$ may not be "even"</u> (where $H(t) \neq H(-t)$, thus $H(\omega)$ would be complex)

$$\Rightarrow \boxed{S_Y(\omega) = \underbrace{H^*(\omega) \cdot H(\omega)}_{|H(\omega)|^2} \cdot S_X(\omega) \text{ or } |H(\omega)|^2 S_X(\omega), \quad \forall \omega \in \mathbb{R}}$$

* NOTE: $R_{XY}(\tau) = (h * R_X)(\tau)$

Def.: a WSS RP is called <u>White Noise</u> if it is zero mean & the autocorrelation is a Dirac Delta function

$$\Rightarrow \boxed{E[X(t)] = 0,\ \forall t \ \wedge\ R_X(\tau) = 0,\ \tau \neq 0}$$

Thus, for white noise, $E[X(t_1)X(t_2)] = 0,\ t_1 \neq t_2$

(for any 2 time instances that are non equal, the expectation of the product for these RVs is zero)

$\therefore\ X(t_1)\ \&\ X(t_2) \leftarrow$ UNCORRELATED (if $t_1 \neq t_2$)
SINCE WHITE NOISE IS ZERO MEAN

* looking at 2 diff. time instances t_1 & t_2, the RVs are uncorrelated

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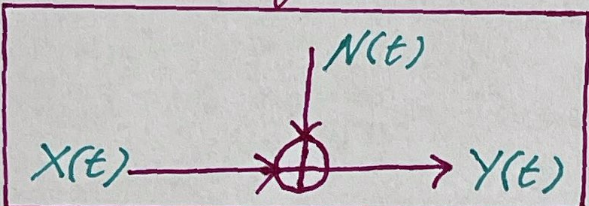
* The Gaussian model & White Noise model are often combined to model noise

Example: WSS Signal, $X(t)$

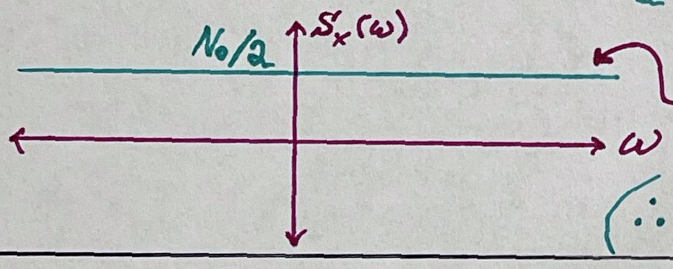


(Often assumed that $N(t)$ is a Gaussian RP & is white noise)

$\Rightarrow$ Additive White Gaussian Noise model (AWGN), where White Noise is an "idealization" → true enough

* What happens when you <u>filter</u> white noise?

* Let $X(t)$ be WN w/ $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$, where <u>$N_0$ is constant</u> & represents the <u>noise power</u>



($S_X(\omega)$, which is constant for all frequencies)

($\therefore$ WN is constant over all frequencies)

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* Let $h(t) = e^{-t}u(t) \leftarrow$ (decaying exponential)
* What happens if WN is passed through filter $h(t)$?

$$\mu_Y = \mu_X \int_{-\infty}^{\infty} h(t)\,dt = 0$$

$$H(\omega) = \frac{1}{1+j\omega} \Rightarrow S_Y(\omega) = |H(\omega)|^2 S_X(\omega) = \frac{N_0/2}{1+\omega^2}$$

$\rightarrow$ (Inverse FT) $R_Y(\tau) = \frac{N_0}{4}e^{-|\tau|},\ \tau \in \mathbb{R}$