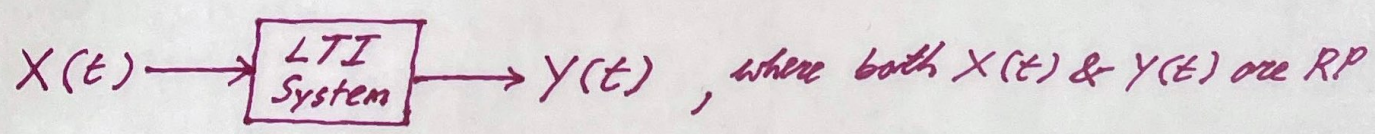


Linear Time Invariant Systems w/ Random Inputs

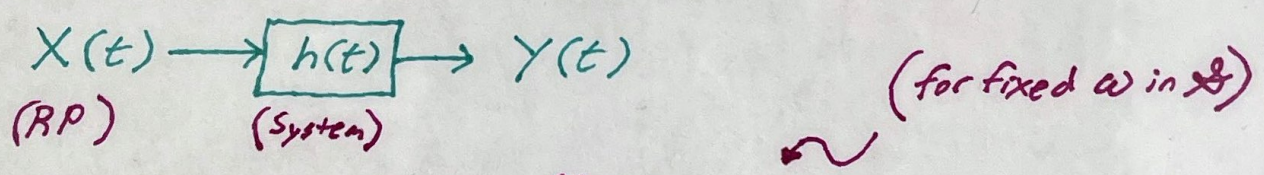


Given μ_x, R_{xx} , & the impulse response of the system, what are μ_y, R_{yy} ? What if $X(t)$ is WSS?

* for an LTI system with deterministic input $\delta(t)$, & impulse response $h(t)$: $\delta(t) \rightarrow$ LTI Sys $\rightarrow h(t)$

* If $x(t)$ is the input, the output is $y(t) = x(t) * h(t)$
↑
 (deterministic signal)

⇒ Now consider:



$$Y(t, \omega) = X(t, \omega) * h(t) = \int_{-\infty}^{\infty} X(t-\alpha, \omega) h(\alpha) d\alpha, \forall \omega \in \mathcal{S}$$

* first find the mean, $\mu_y(t) = E[Y(t)]$:

$E[Y(t)], R_{yy}(t_1, t_2)$?

$$\mu_y(t) = E[Y(t)] = E\left[\int_{-\infty}^{\infty} X(t-\alpha) h(\alpha) d\alpha\right] = \int_{-\infty}^{\infty} \underbrace{E[X(t-\alpha)]}_{\mu_x(t-\alpha)} h(\alpha) d\alpha = \mu_x(t) * h(t)$$

* now the autocorrelation function, $R_{yy}(t_1, t_2)$:

$\mu_y(t) = \mu_x(t) * h(t)$

$$R_{yy}(t_1, t_2) = E[Y(t_1)Y(t_2)] = E\left[\int_{-\infty}^{\infty} X(t_1-\alpha) h(\alpha) d\alpha \int_{-\infty}^{\infty} X(t_2-\beta) h(\beta) d\beta\right] = \iint E[X(t_1-\alpha) X(t_2-\beta)] h(\alpha) h(\beta) d\alpha d\beta$$

$\therefore R_{yy}(t_1, t_2) = \iint_{\mathcal{R}^2} R_{xx}(t_1-\alpha, t_2-\beta) h(\alpha) h(\beta) d\alpha d\beta$

* Very often it is assumed that $X(t)$ is WSS,

in which case:

$$R_{yy}(t_1, t_2) = \iint_{\mathbb{R}^2} R_x(t_2 - t_1 - \rho + \alpha) h(\alpha) h(\rho) d\alpha d\rho$$

$$= \iint_{\mathbb{R}^2} R_x(\tau - \rho + \alpha) h(\alpha) h(\rho) d\alpha d\rho, \tau = t_2 - t_1$$

$$\Rightarrow R_y(\tau) = R_{yy}(t_1, t_2) = \iint_{\mathbb{R}^2} R_x(\tau - \rho + \alpha) h(\alpha) h(\rho) d\alpha d\rho$$

Autocorrelation function for output $Y(t)$ when $X(t)$ is assumed WSS

$$M_y(t) = \int_{-\infty}^{\infty} M_x(t - \alpha) h(\alpha) d\alpha = M_x \int_{-\infty}^{\infty} h(\alpha) d\alpha \leftarrow \text{No dependence on } t$$

$\Rightarrow M_y(t)$ does not depend on t , & the autocorrelation function of Y depends only on the time difference between t_2 & t_1

\therefore If the input to a stable LTI system (is finite: $\int_{-\infty}^{\infty} h(\alpha) d\alpha$) is WSS, then the output is WSS

* Can compact $R_y(\tau)$ as:

$$R_y(\tau) = \int_{-\infty}^{\infty} h(\alpha) \int_{-\infty}^{\infty} h(\rho) R_x(\tau + \alpha - \rho) d\rho d\alpha$$

(Convolution interval
 $(h * R_x)(\tau + \alpha)$
"h convolved w/ R_x
evaluated at $\tau + \alpha$ "

* Let $\lambda = -\alpha$, then

$$R_y(\tau) = \int_{-\infty}^{\infty} h(-\lambda) (h * R_x)(\tau - \lambda) d\lambda$$

($d\lambda$) equals ($-d\alpha$) but when α goes from $(-\infty)$ to (∞) , λ goes from (∞) to $(-\infty)$, thus the minus signs cancel out

$$\Rightarrow R_y(\tau) = (\tilde{h} * h * R_x)(\tau), \text{ where } \tilde{h}(t) = h(-t) \leftarrow \text{Compact form}$$

* the functions R_x & R_y characterize correlations in the processes $X(t)$, $Y(t)$ respectively

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R_x ← describes correlation btwn 2 RVs, $X(t)$ & $X(t+\tau)$

R_y ← describes correlation btwn 2 RVs, $Y(t)$ & $Y(t+\tau)$

What about the correlation between 2 RVs $X(t_1)$ & $Y(t_2)$?

⇒ Cross-Correlation function of RPs $X(t)$ & $Y(t)$: $\tilde{R}_{xy}(t_1, t_2)$

$$\tilde{R}_{xy}(t_1, t_2) = E[X(t_1)Y(t_2)] \leftarrow \text{Cross-Correlation}$$

What about the joint behavior in terms of stationarity?

⇒ RPs $X(t)$ & $Y(t)$ are jointly WSS if each RP is WSS itself

& $\tilde{R}_{xy}(t_1, t_2) = R_{xy}(\tau)$, for some function $R_{xy}: \mathbb{R} \rightarrow \mathbb{R}$, where $\tau = t_2 - t_1$

(*tilda indicates function of 2 RVs)