Linear Time Invariant Systems w/ Random Inputs

$$X(t) \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ \text{System}\end{array}} \longrightarrow Y(t)$$, where both $X(t)$ & $Y(t)$ are RP

Given μ_X, R_{XX}, & the impulse response of the system, what are μ_Y, R_{YY} ? What if $X(t)$ is WSS ?

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* for an LTI system with deterministic input $\delta(t)$, & impulse response $h(t)$: $\delta(t) \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ \text{Sys}\end{array}} \longrightarrow h(t)$
* If $x(t)$ is the input, the output is $y(t) = x(t) * h(t)$

(deterministic signal)

$\Rightarrow$ Now consider:

$$\underset{(RP)}{X(t)} \longrightarrow \underset{(\text{System})}{\boxed{h(t)}} \longrightarrow Y(t)$$

(for fixed ω in $\mathcal{S}$)

$$Y(t,\omega) = X(t,\omega) * h(t) = \int_{-\infty}^{\infty} X(t-\alpha,\omega)\, h(\alpha)\, d\alpha \,,\ \forall \omega \in \mathcal{S}$$

$\boxed{E[Y(t)],\ R_{YY}(t_1,t_2)\ ?}$

* first find the mean, $\mu_Y(t) = E[Y(t)]$:

$$\mu_Y(t) = E[Y(t)] = E\left[\int_{-\infty}^{\infty} X(t-\alpha)\, h(\alpha)\, d\alpha\right] = \int_{-\infty}^{\infty} \underbrace{E[X(t-\alpha)]}_{\mu_X(t-\alpha)}\, h(\alpha)\, d\alpha = \mu_X(t) * h(t)$$

$\therefore \boxed{\mu_Y(t) = \mu_X(t) * h(t)}$

* now the autocorrelation function, $R_{YY}(t_1,t_2)$:

$$R_{YY}(t_1,t_2) = E[Y(t_1)Y(t_2)] = E\left[\int_{-\infty}^{\infty} X(t_1-\alpha)\, h(\alpha)\, d\alpha \int_{-\infty}^{\infty} X(t_2-\beta)\, h(\beta)\, d\beta\right] = \iint E[X(t_1-\alpha) \cdot X(t_2-\beta)] \cdot h(\alpha)\, h(\beta)\, d\alpha\, d\beta$$

$$\therefore \boxed{R_{YY}(t_1,t_2) = \iint_{\mathbb{R}^2} R_{XX}(t_1-\alpha,\, t_2-\beta)\, h(\alpha)\, h(\beta)\, d\alpha\, d\beta}$$

* Very often it is assumed that $X(t)$ is WSS,

in which case: $$R_{YY}(t_1,t_2) = \iint_{\mathbb{R}^2} R_X(t_2-t_1-\beta+\alpha)\,h(\alpha)\,h(\beta)\,d\alpha\,d\beta$$

$$= \iint_{\mathbb{R}^2} R_X(\tau-\beta+\alpha)\,h(\alpha)\,h(\beta)\,d\alpha\,d\beta, \quad \tau = t_2-t_1$$

$$\Rightarrow R_Y(\tau) = R_{YY}(t_1,t_2) = \iint_{\mathbb{R}^2} R_X(\tau-\beta+\alpha)\,h(\alpha)\,h(\beta)\,d\alpha\,d\beta$$

Autocorrelation function for output $Y(t)$ when $X(t)$ is assumed WSS

$$\mu_Y(t) = \int_{-\infty}^{\infty} \mu_X(t-\alpha)\,h(\alpha)\,d\alpha = \mu_X \int_{-\infty}^{\infty} h(\alpha)\,d\alpha$$ ← No dependance on t

$\Rightarrow$ $\mu_Y(t)$ does not depend on t, & the autocorrelation function of Y depends only on the time difference between t_2 & t_1

$\therefore$ If the input to a stable LTI system (is finite: $\int_{-\infty}^{\infty} h(\alpha)\,d\alpha$) is WSS, then the output is WSS

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* Can compact $R_Y(\tau)$ as:

$$R_Y(\tau) = \int_{-\infty}^{\infty} h(\alpha) \underbrace{\int_{-\infty}^{\infty} h(\beta)\,R_X(\tau+\alpha-\beta)\,d\beta}_{\substack{\text{Convolution integral} \\ (h * R_X)(\tau+\alpha) \\ \text{"h convolved w/ } R_X \\ \text{evaluated at } \tau+\alpha\text{"}}}\,d\alpha$$

*Let $\lambda = -\alpha$, then

$$R_Y(\tau) = \int_{-\infty}^{\infty} h(-\lambda)\,(h * R_X)(\tau-\lambda)\,d\lambda$$

($d\lambda$) equals ($-d\alpha$) but when α goes from ($-\infty$) to (∞), λ goes from (∞) to ($-\infty$), thus the minus signs cancel out

$$\Rightarrow \boxed{R_Y(\tau) = (\tilde{h} * h * R_X)(\tau), \text{ where } \tilde{h}(t) = h(-t)}$$ ← Compact form

* the functions R_X & R_Y characterize correlations in the processes $X(t)$, $Y(t)$ respectiveley

R_X ← describes correlation btwn 2 RVs, $X(t)$ & $X(t+\tau)$

R_Y ← describes correlation btwn 2 RVs, $Y(t)$ & $Y(t+\tau)$

What about the correlation between 2 RVs $X(t_1)$ & $Y(t_2)$?

⇒ Cross-Correlation function of RPs $X(t)$ & $Y(t)$: $\tilde{R}_{XY}(t_1, t_2)$

$$\boxed{\tilde{R}_{XY}(t_1, t_2) = E[X(t_1)\,Y(t_2)] \leftarrow \text{Cross-Correlation}}$$

What about the joint behavior in terms of stationarity ?

⇒ RPs $X(t)$ & $Y(t)$ are jointly WSS if each RP is WSS itself

& $\tilde{R}_{XY}(t_1, t_2) = R_{XY}(\tau)$, for some function $R_{XY} : \mathbb{R} \to \mathbb{R}$, where $\tau = t_2 - t_1$

(*tilda indicates function of 2 RVs)