Stationarity

In practice it can be difficult to model & apply pdfs, mean function, autocorrelation & autocovariance functions of a random process $X(t)$ } Common approach in practice is to assume some form of "stationarity"

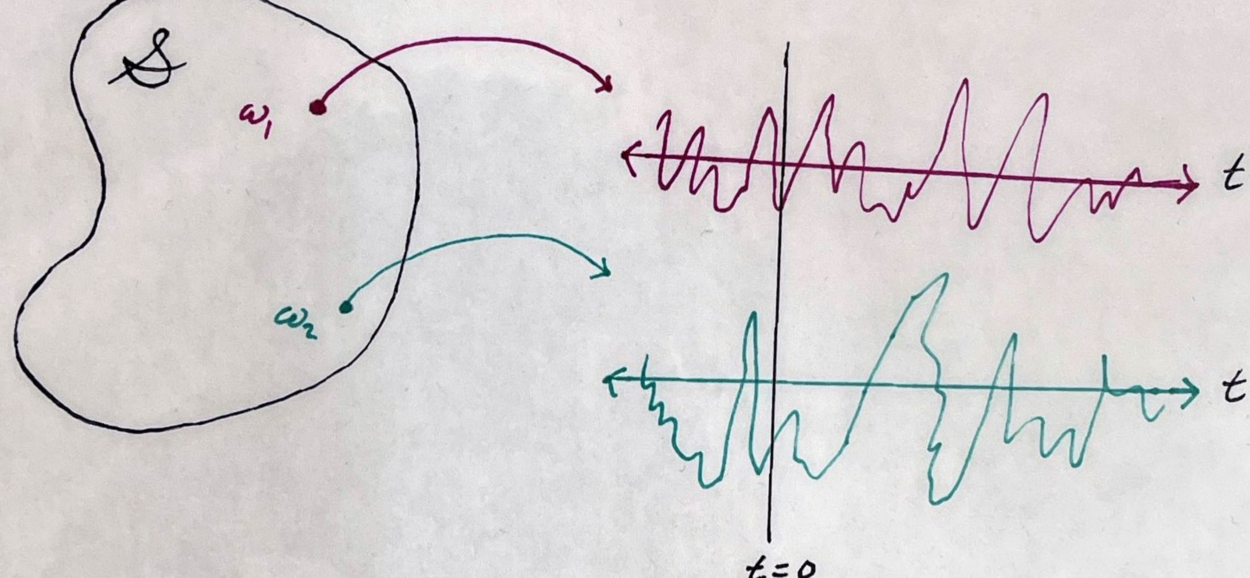
Example: Estimate $\mu_X(t)$ & $R_{XX}(t_1, t_2)$ via 3 samples: $\hat{X}(t_1), \hat{X}(t_2), \hat{X}(t_3)$

* having 1 sample each for $\mu_X(t_1), \mu_X(t_2), \mu_X(t_3)$

$\Rightarrow$ Let $\hat{\mu}_X(t_1) = \hat{X}(t_1)$, $\hat{\mu}_X(t_2) = \hat{X}(t_2)$, $\hat{\mu}_X(t_3) = \hat{X}(t_3)$

* having 1 sample each for $R_{XX}(t_1, t_2)$, $R_{XX}(t_2, t_3)$

$\Rightarrow$ Let $\hat{R}_{XX}(t_1, t_2) = \hat{X}(t_1)\hat{X}(t_2)$, $\hat{R}_{XX}(t_2, t_3) = \hat{X}(t_2)\hat{X}(t_3)$



* Do these functions depend on absolute or relative time?

$\Rightarrow$ Stationarity addresses whether or not the distribution of a function depends on where the origin is (t_0)

Def.: Strict-Sense Stationary (SSS) Random Process

a RP $X(t)$ is SSS if (assumed true in practice)

$$f_{X(t_1)\dots X(t_n)}(x_1,\dots,x_n) = f_{X(t_1+\alpha)\dots X(t_n+\alpha)}(x_1,\dots,x_n)$$

$$\left(\begin{array}{l}\forall \alpha \in \mathbb{R}, n \in \mathbb{N};\ t_1,\dots,t_n \in T \\ \wedge\ (x_1,\dots,x_n)^T \in \mathbb{R}^n\end{array}\right)$$ (time index set)

Does shifting the time origin by alpha lead to the same n^{th} order density function?

If so $\Rightarrow$ Process is SSS

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* If $X(t)$ is SSS, then $f_{X(t)}(x) = f_{X(t+\alpha)}(x) = f_X(x)$, $\forall t, \alpha, x \in \mathbb{R}$, & for some 1^{st} order pdf f_X

* If $X(t)$ is SSS $f_{X(t_1)X(t_2)}(x_1,x_2) = f_{X(t_1+\alpha)X(t_2+\alpha)}(x_1,x_2) = f_{X_1X_2}(x_1,x_2,\tau)$

where ($\tau = t_2 - t_1$ $\wedge$ $f_{X_1X_2}$ is a 2^{nd} order pdf)

Def.: Wide-Sense Stationary (WSS) Random Process

$X(t)$ SSS $\Rightarrow X(t)$ WSS
$X(t)$ WSS $\not\Rightarrow X(t)$ SSS

a RP $X(t)$ is WSS if

(i) $E[X(t)] \equiv \mu_X(t) = \mu_X$, $\forall t$ for some $\mu_X \in \mathbb{R}$

(ii) $R_{XX}(t_1,t_2) = R_X(t_2 - t_1) = R_X(\tau)$, $\forall t_1, t_2 \in T$, where $\tau = t_2 - t_1$ $\wedge$ $R_X : \mathbb{R} \to \mathbb{R}$

* If $X(t)$ is WSS, then $E[X^2(t)] = R_{XX}(t,t) = R_X(0)$, so $R_X(0) \geq 0$

* $C_{XX}(t_1,t_2) = C_X(\tau)$, $\tau = t_2 - t_1$ for some $C_X : \mathbb{R} \to \mathbb{R}$

$\Rightarrow C_X(\tau) = R_X(\tau) - \mu_X^2$ $\leftarrow$ (mean is constant w.r.t. time for WSS RP)

* If $X(t)$ is WSS & is Guassian, then $X(t)$ is SSS

Proof for "If $X(t)$ is WSS & Gaussian, then $X(t)$ is SSS":

The RVS $X(t_1+\alpha), \ldots, X(t_n+\alpha)$ have the char. fn:

$$\Phi_{X(t_1+\alpha)\cdots X(t_n+\alpha)}(\omega_1, \ldots, \omega_n) = \exp\left[i\mu_X \sum_{k=1}^{n} \omega_k - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} C_X\underbrace{(t_k+\alpha-(t_j+\alpha))}_{(t_k-t_j)}\omega_j\omega_k\right]$$

* Since there is no dependence on α, $F_{X(t_1+\alpha)\cdots X(t_n+\alpha)}$ does not depend on α $\Rightarrow \therefore X(t)$ is SSS

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* If $X(t)$ is assumed to be WSS & there are 3 samples $\hat{X}(t_1), \hat{X}(t_2), \hat{X}(t_3)$, then the following may be used:

$$\hat{\mu}_X = \frac{1}{3}\sum_{i=1}^{3} X(t_i)$$

$$\hat{R}_X(1) = \frac{1}{2}\left[\hat{X}(t_1)\hat{X}(t_2) + \hat{X}(t_2)\hat{X}(t_3)\right]$$

$$\hat{R}_X(2) = \hat{X}(t_1)\hat{X}(t_3)$$

* Sometimes a RP $X(t)$ is obviously Non-Stationary (NS), but there may be interest in the "difference" sequence: $Y(t) \equiv X(t) - X(t-1)$ which might be modeled as stationary in some sense (adaptive filtering)

$Y(t) \equiv X(t) - X(t-1)$ ← Predict $X(t)$ after observing $X(t-1)$, meaning $Y(t)$ is the prediction error