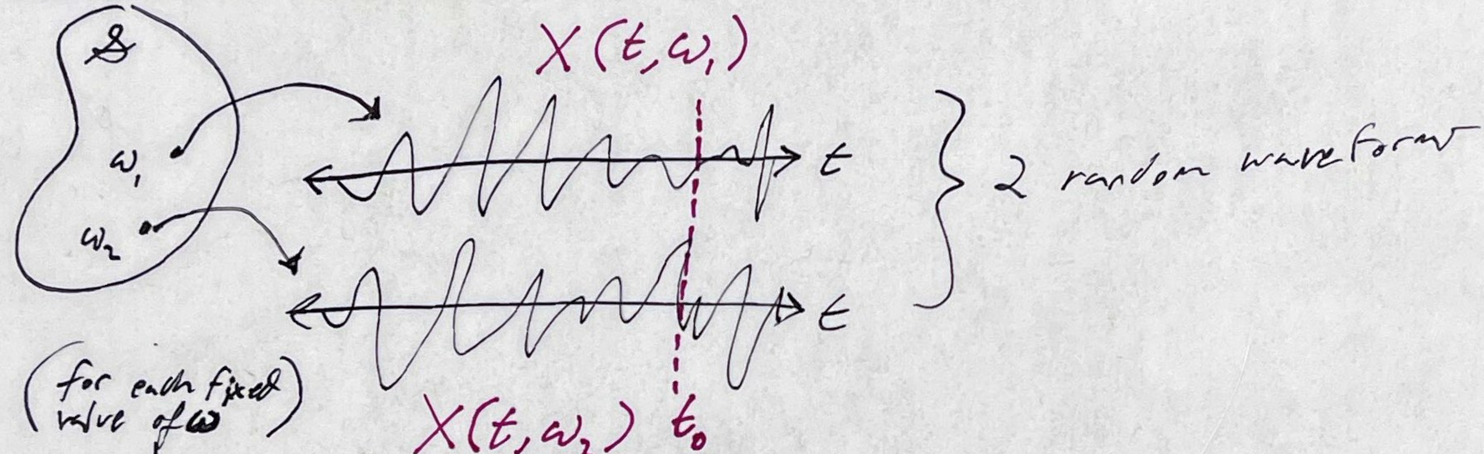
Stochastic / Random Processes (rp)

* Now adding "time" variable to system of RVs.
* Random sequences will be a special case of this new case of Random Variables indexed by time. (time is discretized)

Def. of a Stochastic Process: defined on $(\mathcal{S}, \mathcal{F}, P)$ is a family/group/collection of random variables $\{X(t), t \in T\}$ indexed by a set T



Each waveform is referred to as a sample function or a sample realization (any type of RV), sample function is more specific to random processes (b/c when you have a sample of time when you fix ω)

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* Note that $X(t, \omega)$, or simply $X(t)$ is a Random Process
* $X(t_0, \omega)$ is a rv for a fixed $t_0 \in T$
* $X(t, \omega_0)$ is a real-valued function of t for fixed $\omega_0 \in \mathcal{S}$
* $X(t_0, \omega_0)$ is a real number for fixed t_0, ω_0

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4 Types of Random Processes

1) $T = \mathbb{R}$, or $\mathbb{R}^+$, or an interval in $\mathbb{R}$: Continuous-Time rp, with each $X(t)$ being a continuous RV.

2) $T = \mathbb{R}, \mathbb{R}^+$, or an interval in $\mathbb{R}$, each $X(t)$ is a discrete RV: continuous-time discrete-valued rp

3) $T = \mathbb{N}$, or another countably infinite subset of $\mathbb{R}$, & each $X(t)$ a continuous RV: discrete-time continuous-valued rp

4) Discrete-time ($T = \mathbb{N}$), each $X(t)$ a discrete RV

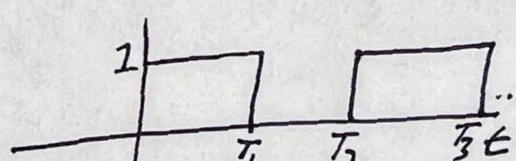
Examples:

1) $T = \mathbb{N}$ & $X(t)$ is a gaussian RV for each $t \in \mathbb{N}$

(Note that if $t = n$, usually write the rp as $X(1), X(2), \ldots,$ or $X_1, X_2, \ldots$)

Discrete-time, continuous-valued rp

2) Binary waveform w/ random transition times



(points T_1, T_2, T_3 are Random variables)

Continuous-time, Discrete-valued rp

(where $T = \mathbb{R}^+$)

(could also represent this as a random sequence $T_1, T_2, \ldots$ & a starting value ± 1)

3) Sinusoid w/ random frequency

$X(t) = \sin(\Omega t)$, where Ω is a continuous Random Variable on $[0, 2\pi]$

Continuous-time continuous rp

Probabolistic Dercription of a rp

* Use probability distributions to characterize random processes

Def. of the n^{th} order CDF of rp $X(t)$:

$$F_{X(t_1),\dots X(t_n)}(x_1,\dots,x_n) = P(X(t_1)\le x_1,\dots,X(t_n)\le x_n)$$

for $n\in\mathbb{N}$ & $(x_1,\dots,x_n)\in\mathbb{R}^n$,
& all $t_1 < t_2 < \dots < t_n$,
where each $t_i \in T$

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* note that for $n=1$, $f_{X(t_1)}(x)$ as a 1^{st} order pdf

* For the case $n=2$, ~~$f_{X(t_1)}$~~
$f_{X(t_1)X(t_2)}(x_1,x_2)$ as a 2^{nd} order pdf

Def.: Mean (function) of a rp $X(t)$, $\mu_X(t) \equiv E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)\,dx$,

$\forall t \in T$

Def.: Autocorrelation function of a rp $X(t)$, $R_{XX}(t_1,t_2) \equiv E[X(t_1)X(t_2)]$

$$= \iint_{\mathbb{R}^2} x_1 x_2 f_{X(t_1)X(t_2)}(x_1,x_2)\,dx_1\,dx_2$$

$\forall t_1, t_2 \in T$

Note: $R_{XX}(t_2,t_1) = R_{XX}(t_1,t_2)$

Def.: Autocovariance function of a rp $X(t)$,

$$C_{XX}(t_1,t_2) \equiv E[(X(t_1)-\mu_X(t_1))(X(t_2)-\mu_X(t_2))] = R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)$$

* Note that R_{XX} & C_{XX} are Non-Negative Definite (NND) functions, i.e., for $\forall (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\forall n \in \mathbb{N}$, $\forall t_1, \ldots, t_n$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j R_{XX}(t_i, t_j) \geq 0$$

Def. of ~~an~~ 1 Important Random Process: <u>Gaussian Random Process</u>

$X(t)$ is a rp satisfying the property that

$X(t_1), \ldots, X(t_n)$ are jointly Gaussian RVs

$\forall n \in \mathbb{N}$ & every $t_1 < t_2 < \ldots < t_n$

$t_i \in T$

the n^{th} order pdfs in this case are those of Gaussian random vectors & depend only on $\mu_X(t)$ & $C_{XX}(t_1, t_2)$