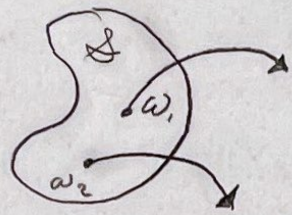
Random Vectors

(n RVs defined on a space $(\mathcal{S}, \mathcal{F}, P)$ for some finite n)

* Def.: Let $X_1, \ldots, X_n$ be n RVs on $(\mathcal{S}, \mathcal{F}, P)$ for some finite $n \geq 1$

$$\underline{X} = [X_1 \ \ldots \ X_n]^T \longleftarrow \text{Rand. Vec. on } (\mathcal{S}, \mathcal{F}, P)$$

(Random Column Vector)



$$\underline{X}(\omega_1) = \begin{bmatrix} X_1(\omega_1) \\ \vdots \\ X_n(\omega_1) \end{bmatrix}$$

$$\underline{X}(\omega_2) = \begin{bmatrix} X_1(\omega_2) \\ \vdots \\ X_n(\omega_2) \end{bmatrix}$$

* Most definitions & properties for 2 RVs extend naturally to the case of n RVs

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* The n^{th} order density fn of a Rand. Vec.:

$$\boxed{f_{\underline{X}}(\underline{x}) = \frac{\partial^n}{\partial x_1 \ldots \partial x_n} F_{\underline{X}}(\underline{x}), \ \forall \underline{x} = [x_1, \ldots, x_n]^T \in \mathbb{R}^n}$$

* For any $D \in B(\mathbb{R}^n)$, $P(X \in D) = \int_D f_{\underline{X}}(\underline{x}) d\underline{x}$

(where $B(\mathbb{R}^n)$ is the sigma field generated by the set of n-dim hypercubes in $\mathbb{R}^n$)

* Components of $\underline{X}$ are STATISTICALLY INDEPENDENT if:

$$\boxed{f_{\underline{X}}(\underline{x}) = \prod_{i=1}^{n} f_{X_i}(x_i), \ \forall \underline{x} \in \mathbb{R}^n}$$

(for all vectors $\underline{x}$ in $\mathbb{R}^n$)

Random Vectors : Moments

* Often characterize a random vector via 1st & 2nd order moments

* Correlation btwn X_j & X_k is denoted R_{jk}

$$\Rightarrow R_{jk} \equiv E[X_j X_k], \quad (j,k) = 1, \dots, n$$

if $j=k$, $R_{jj} = R_{kk} = E[X_j^2] = E[X_k^2]$

* Covariance of X_j & X_k is denoted C_{jk}

$$\Rightarrow C_{jk} \equiv E[(X_j - \bar{X}_j)(X_k - \bar{X}_k)]$$

if $j=k$, $C_{jj} = C_{kk} = \sigma_{X_j}^2 = \sigma_{X_k}^2$

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3 MOMENTS USED 2 CHARACTERIZE A RANDOM VECTOR

* Def. : Correlation Matrix for $\underline{X}$ is denoted $R_{\underline{X}}$

1 $\Rightarrow$
$$\boxed{R_{\underline{X}} = \begin{bmatrix} R_{11} & \cdots & R_{1n} \\ \vdots & & \vdots \\ R_{n1} & \cdots & R_{nn} \end{bmatrix}}$$
$\leftarrow$ (Diag. contains mean-squared values of X_1 to X_n, off-diagonal terms are the correlations of 2 RVs)

* Def. : Covariance Matrix for $\underline{X}$ is denoted $C_{\underline{X}}$

2 $\Rightarrow$
$$\boxed{C_{\underline{X}} = \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & & \vdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix}}$$
$\leftarrow$ (Diag. terms are the variances of X_1 to X_n, off-diagonal terms are the co-variances)

* Def. : Mean Vector of $\underline{X}$ is denoted $\mu_{\underline{X}}$

3 $\Rightarrow$
$$\boxed{\mu_{\underline{X}} = [E[X_1], \dots, E[X_n]]^T}$$

$\therefore$

$$R_{\underline{X}} = E[\underbrace{\underline{X}\,\underline{X}^T}_{(n\times1)(1\times n)}] \leftarrow (n \times n)$$

$$C_{\underline{X}} = E[\underbrace{(\underline{X} - \mu_{\underline{X}})}_{(n\times1)}\underbrace{(\underline{X} - \mu_{\underline{X}})^T}_{(1\times n)}] = (n \times n)$$

Random Vectors ; Properties (of $R_{\underline{X}}$ & $C_{\underline{X}}$)

* Def.: Let b_{ij} be the i, j th element of an $n \times n$ matrix, is non-negative definite (NND) if:

$$\boxed{\sum_{i=1} \sum_{j=1} x_i x_j - b_{ij} \geq 0, \ \forall \text{ real vector } \underline{x} \in \mathbb{R}^n}$$

* Note that for "random processes", $R_{\underline{X}} \leftarrow$ Autocorrelation Matrix

* For any rand. vec. $\underline{X}$, its correlation matrix is NND

$\Rightarrow$ Proof: Let $\underline{x}$ be an arbitrary vector in $\mathbb{R}^n$ &

$$Y = \underline{x}^T \underline{X} = \underbrace{\underline{X}^T}_{(1\times n)} \underbrace{\underline{x}}_{(n\times 1)} \leftarrow (Y \text{ is a scalar rv})$$

$= 1 \times 1$ scalar

$$\Rightarrow 0 \leq E[Y^2] = E[\underline{x}^T \underline{X} \underline{X}^T \underline{x}] = \underline{x}^T E[\underline{X}\underline{X}^T] \underline{x}$$

$$= \underline{x}^T R_{\underline{X}} \underline{x}$$

$$0 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j R_{ij} \quad \therefore R_{\underline{X}} \text{ is NND}$$

(can show same result for $C_{\underline{X}}$)

Gaussian Random Vectors:

* Def.: a Rand. Vec. $\underline{X}$ is gaussian if: $Z = a_0 + \sum_{j=1}^{n} a_j X_j$ is a gaussian RV $\forall [a_0, \ldots, a_n]^T \in \mathbb{R}^{n+1}$

* Instead of deriving a density $f_{\underline{X}}(\underline{x})$ for a gaussian Rand. Vec., first find the characteristic function of $\underline{X}$

Characteristic Function of a Random Vector $\underline{X}$ (we 2 find pdfs of sums of RVs)

$$\boxed{\Phi_{\underline{X}}(\underline{\Omega}) = E\left[e^{i \sum_{j=1}^{n} \omega_j X_j}\right], \text{ for } \underline{\Omega} = [\omega_1, \ldots, \omega_n]^T \in \mathbb{R}^n}$$

* If $Z = \sum_{j=1}^{n} X_j$, then $\Phi_Z = E\left[e^{i\omega \sum_{j=1}^{n} X_j}\right] = \Phi_{\underline{X}}(\omega, \ldots, \omega)$ ($\because$ every $\omega_j = \omega$)

* If $X_1, \ldots, X_n$ are independent then $\Phi_Z(\omega) = \prod_{j=1}^{n} \Phi_{X_j}(\omega)$

* If $X_1, \ldots, X_n$ are i.i.d., then $\Phi_Z(\omega) = (\Phi_X(\omega))^n$ (where Φ_X is the common char. fn of $X_1, \ldots, X_n$)

Char. Fn of Gaussian Rand. Vec. $\underline{X}$

$$\boxed{\Phi_{\underline{X}}(\underline{\Omega}) = e^{i\underline{\Omega}^T\mu_{\underline{X}} - \frac{1}{2}\underline{\Omega}^T C_{\underline{X}}\underline{\Omega}}} \leftarrow$$

(Char. fn in terms of the mean-vector μ_X & covariance fn C_X (2 moments))

Proof: Let $Z = \sum_{j=1}^{n} \omega_j X_j$, for $\Omega \in \mathbb{R}^n$

then Z is a Gaussian RV $\because$ $\underline{X}$ is a Gaussian Rand. Vector

$$\Rightarrow \Phi_Z(\omega) = \left(e^{i\omega\mu_Z}\right)\left(e^{-\frac{1}{2}\omega^2\sigma_Z^2}\right),$$ where

$$\mu_Z = E[Z] = \sum_{j=1}^{n} \omega_j \mu_{X_j} = \underline{\Omega}^T \mu_{\underline{X}}$$

$$\sigma_Z^2 = Var(Z) = \sum_{j=1}^{n}\sum_{k=1}^{n} \sigma_{jk}\,\omega_j\omega_k = \underline{\Omega}^T C_{\underline{X}}\underline{\Omega}$$

(if $j=k$)

$$\sigma_{jk} = E[(X_j - \overline{X}_j)(X_k - \overline{X}_k)]$$

* Z is a linear combination (affine) of $X_1, \dots, X_n$, which is gaussian (since X is a gaussian random vector)

$$\Rightarrow \Phi_{\underline{X}}(\underline{\Omega}) = E\left[e^{i\sum_{j=1}^{n}\omega_j X_j}\right] \leftarrow \text{(Char. fn of a Random Vector)}$$

$$= \Phi_Z(\omega)$$

* Substitute $\mu_Z \wedge \sigma_Z^2$ in $\Phi_Z \Rightarrow$ $\boxed{\Phi_{\underline{X}}(\underline{\Omega}) = \left(e^{i\underline{\Omega}^T\mu_{\underline{X}}}\right)\left(e^{-\frac{1}{2}\underline{\Omega}^T C_{\underline{X}}\underline{\Omega}}\right)}$

$\Rightarrow$ Density function of a gaussian random vector $\underline{X}$ is:

(pdf)

$$\boxed{f_X(\underline{x}) = \frac{1}{\sqrt{(2\pi)^n |C_{\underline{X}}|}} \exp\left[-\frac{1}{2}\underbrace{(\underline{x} - \mu_{\underline{X}})^T C_{\underline{X}}^{-1}(\underline{x} - \mu_{\underline{X}})}_{(\geq 0) \because C_{\underline{X}}^{-1} \text{ is NND}}\right]}$$

* In practice, common to seek the most likely value of $\underline{x}$ for a problem which requires maximizing this density function (via maximizing the natural log of the density function)

$\Rightarrow$ Results in a quadratic in the vector x containing a unique minimum ($\because$ NND Property of C_X^{-1})

(Maximization of the likelihood via minimizing the log of this quadratic)