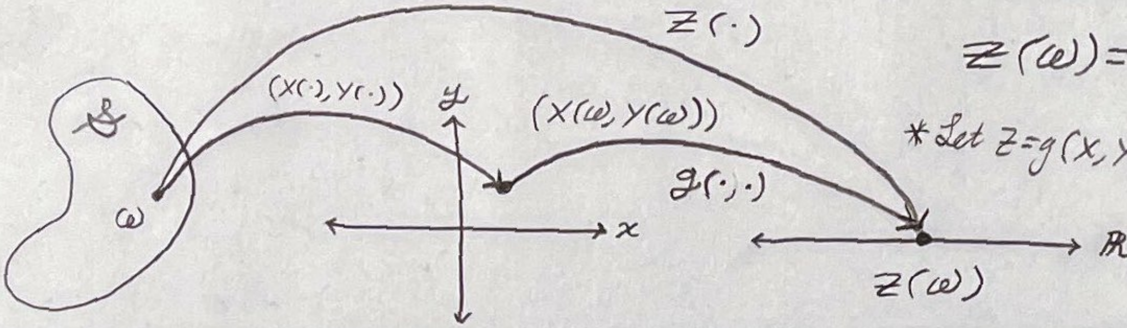
One Function of 2 Random Variables

given 2 RVs X & Y on prob. space $(\mathcal{S}, \mathcal{F}, P)$



$$Z(\omega) = g(X(\omega), Y(\omega))$$

* Let $Z = g(X, Y)$

g is a function mapping $\mathbb{R}^2$ to $\mathbb{R}$
$g: \mathbb{R}^2 \to \mathbb{R}$

---

* Assume g meets requirements for Z to be a valid RV
* Find the density function of Z, $\forall z \in \mathbb{R}$, by starting with the cdf (probability)

$$\underbrace{P(Z \leq z)}_{(=F_Z(z))} = P(g(X(\omega), Y(\omega)) \leq z) = P((X(\omega), Y(\omega)) \in D_z)$$

where $D_z = \{(x, y) \in \mathbb{R}^2 : g(x, y) \leq z\}$

---

* In order to say that the RV Z is in some region of the real line with some probability, first find the set omega in $\mathcal{S}$ that is mapped by $g(X(\omega))$ to that region in the real line
* With the assumption of Z being a RV, we know $\{(x,y) \in D_z\} \in \mathcal{F}$
this event is $\{(x,y) \in D_z\} = \{\omega \in \mathcal{S} : (X(\omega), Y(\omega)) \in D_z\}$

$\Rightarrow$ After finding D_z for $z \in \mathbb{R}$, let $F_Z(z) = \iint_{D_z} f_{XY}(x,y)\,dx\,dy$

(joint density of X & Y)

* find D_z for each fixed z in $\mathbb{R}$ (depends on $g(\cdot)$)
* then find the cdf of Z at z via integrating the joint density of X & Y over D_z

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

In practice we would know g, & find d_z for each z & then differentiate

* Conceptually — This is fundamental to how Random variables are linked to the prob. space. $(\mathcal{S}, \mathcal{F}, P)$

1 function of 2 RVs — Example

*** THE DISTRIBUTION FUNCTION IS THE INTEGRAL OF THE DENSITY FUNCTION**

Let $\underbrace{g(x,y) = x+y}$, then $Z = X+Y$

(functional form, not random fn until RV X replaces x & RV Y replaces y)

$\Rightarrow$ $\boxed{Z = g(X,Y) = X+Y}$

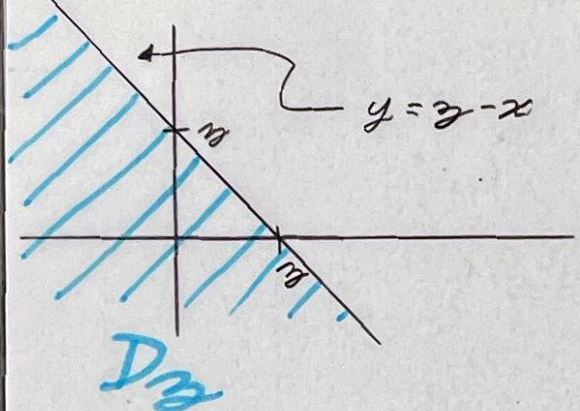
* now find the density function for the random variable $X+Y$, $f_Z(z)$

Find $f_Z(z)$

* for fixed $z \in \mathbb{R}$, $F_Z(z) = P(X+Y \le z) = P((X,Y) \in D_z)$

where $D_z = \{(x,y) \in \mathbb{R}^2 : x+y \le z\}$

$= \{(x,y) : y \le z - x\}$



$$\Rightarrow \boxed{F_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_{XY}(x,y)\,dy\,dx}$$

(Integrate the joint density of X & Y, over the region of D_z via integration)

(CDF solution)

Find $F_Z(z)$ — iff X & Y are ind., then $F_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_X(x) f_Y(y)\,dy\,dx$

$$= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{z-x} f_Y(y)\,dy\,dx = \int_{-\infty}^{\infty} f_X(x) F_Y(z-x)\,dx$$

$\leftarrow$ (form of cdf for $Z = X+Y$ where X & Y are ind.)

*** RECALL THAT THE DENSITY FUNCTION IS THE DERIVATIVE OF THE CDF**

$$\Rightarrow f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} f_X(x) \frac{d}{dz} F_Y(z-x)\,dx$$

(pdf)

$$= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$$

Adding 2 ind. RVs $\Longleftrightarrow$ Convolving their density functions

$$\Rightarrow \therefore f_Z(z) = (f_X * f_Y)(z)$$

("f_X convolved with f_Y of z")

1 function of 2 RVs — Example 2

iff X & Y are exp. RVs with means $\mu_X = \mu_Y = \mu$,
X & Y are ind. & $Z = X + Y$ $f_Z(z) = (f_X * f_Y)(z)$

$$\begin{cases} f_Z(z) = \dfrac{z}{\mu^2} \exp(-z/\mu)\, u(z), & \text{for } z \geq 0 \\ f_Z(z) = 0, & \text{for } z < 0 \end{cases}$$

* The sum of 2 ind. exp. RVs is NOT itself a random variable