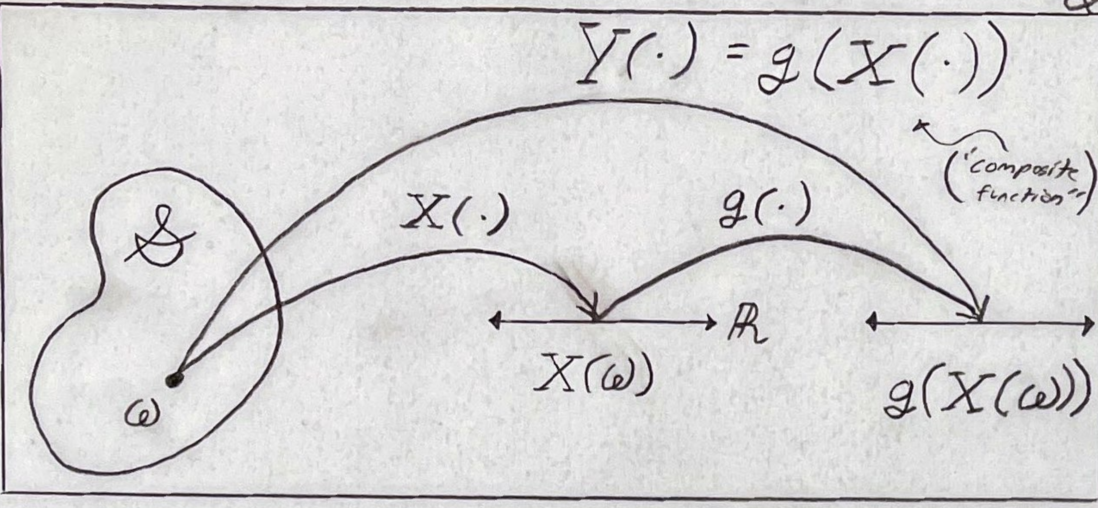
Functions of a RV: Let X be a RV on probability space: $(\mathcal{S}, \mathcal{F}, P)$, & consider a mapping



$g: \mathbb{R} \mapsto \mathbb{R}$ ...

Then it may be said that $Y(\cdot)$ is a composit function, mapping $\mathcal{S}$ to $\mathbb{R}$

* Question: Is Y a RV?

$\Rightarrow Y(\omega) = g(X(\omega)), \ \forall \omega \in \mathcal{S}$

Required Properties for a RV (2):

(Event space where $\mathcal{S}$ is the sample space)

(i) Y is a function mapping $\mathcal{S}$ to $\mathbb{R}$ $\iff Y: \mathcal{S} \mapsto \mathbb{R}$

(ii) Y is a Borel-Measurable Function $\iff \exists\, Y^{-1}(A) = \{\omega \in \mathcal{S} : Y(\omega) \in A\} \in \mathcal{F}, \ \forall A \in \mathcal{B}(\mathbb{R})$

Since $X: \mathcal{S} \mapsto \mathbb{R}$, $g: \mathbb{R} \mapsto \mathbb{R}$, the composit function $(X \circ g)$ can be said to map $\mathcal{S}$ to $\mathbb{R}$ for all ω in $\mathcal{S}$, such that:

$$\left.\begin{aligned} &Y(\omega) = g(X(\omega)), \ \forall \omega \in \mathcal{S} \\ &\iff \\ &Y(\omega) \equiv (g \circ X)(\omega) \equiv g(X(\omega)) \end{aligned}\right\} \quad \begin{aligned} &(g \circ X): \mathcal{S} \mapsto \mathbb{R} \\ &\iff \\ &Y: \mathcal{S} \mapsto \mathbb{R} \end{aligned}$$

Now show that $\exists\, Y^{-1}(A) = \{\omega \in \mathcal{S} : Y(\omega) \in A\} \subset \mathcal{F}$ (event space in the prob. space where $\mathcal{S}$ is the sample space)

$= \{\omega \in \mathcal{S} : g(X(\omega)) \in A\} \subset \mathcal{F}$

$$\left.\text{The inverse image of } A \text{ is a proper subset in } \mathcal{S} \text{ for all } A \text{ in } \mathcal{B}(\mathbb{R})\right\} \ Y^{-1}(A) \subset \mathcal{S}, \ \forall A \in \mathcal{B}(\mathbb{R})$$

$\Rightarrow$ Restrict g to satisfy (ii), in order to consider Y a RV

↳ what is the distribution of Y ? (3 Cases) (2)

(knowing that $Y = g(X(\cdot))$) for RV

- X, Y both contin. RVs
- X, Y both discrete RVs
- X continuous, Y discrete

Case-1) X, Y ← Continuous RVs

looking for the pdf of Y

first, find the cdf of Y, $F_Y(y)$, then differentiate to get $f_Y(y)$

$\Rightarrow \boxed{F_Y(y) = P(g(X) \leq y) = P(X \in D_y)}$, where $D_y \equiv \{x \in \mathbb{R} : g(x) \leq y\} \subset \mathcal{B}(\mathbb{R})$

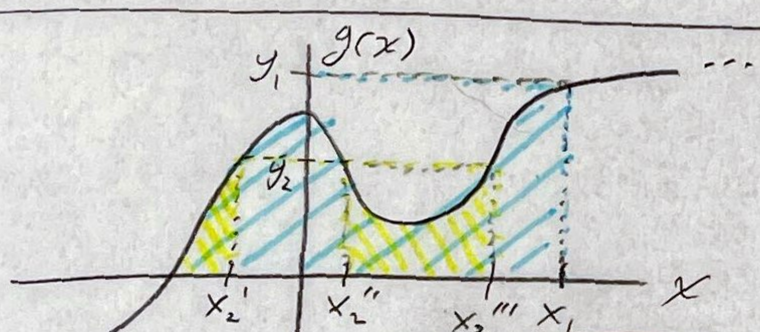
D_y may also be written as:

$D_y = g^{-1}((-\infty, y])$

(in form of an inverse image)

$\boxed{F_Y(y) = \int_{D_y} f_X(x)\,dx, \ \forall y \in \mathbb{R}}$

($\Rightarrow$ differentiate F_Y to get f_Y)



$*$ density fn $f_Y(y) = \frac{d}{dy} F_Y(y)$

Find D_y for $y = y_1$ & $y = y_2$

(1 solution $y_1 = g(x_1)$) (3 solutions)

$\{Y \leq y_1\} = \{X \in D_{y_1}\}$

$D_{y_1} = \{x : x \leq x_1\}$ only depends on $g(\cdot)$

$D_{y_2} = \{x : x \leq x_2'\} \cup \{x : x_2'' < x \leq x_2'''\}$

$P(\{Y \leq y_1\}) = P(\{X \in D_{y_1}\})$

$= P(\{X \leq x_1\}) \Rightarrow \boxed{D_{y_1} = (-\infty, x_1]}$

"the set of values in omega that the rv Y maps to is a point below y_2 is the same as the set of omega in & that X maps to one of the areas in the horizontal axis $\leq x_2'$ & $x_2'' < x \leq x_2'''$"

$D_{y_2}: \{Y \leq y_2\} = \{X \leq x_2'\} \cup \{x_2'' \leq X \leq x_2'''\}$

$\boxed{D_{y_2} = (-\infty, x_2'] \cup [x_2'', x_2''']}$

$\therefore F_Y(y_1) = \int_{D_{y_1}} f_X(x)\,dx = \int_{-\infty}^{x_1} f_X(x)\,dx$

$F_Y(y_2) = \int_{-\infty}^{x_2'} f_X(x)\,dx + \int_{x_2''}^{x_2'''} f_X(x)\,dx$

Example: $Y = aX + b$; $a, b \in \mathbb{R}$; $a \neq 0$

then $P(Y \le y) = P(aX + b \le y) = P\left(X \le \frac{y-b}{a}\right)$

$= F_X\left(\frac{y-b}{a}\right) \ \forall y \in \mathbb{R}$

Approach 2: Change of Variables

inverse function (not image)

If $Y = g(X)$ it can be shown that $f_Y(y) = \dfrac{f_X(g^{-1}(y))}{\left|\frac{dy}{dx}\right|_{x = g^{-1}(y)}}$

... as long as:

(i) exists a unique solution

& $x = g^{-1}(y) \ \forall y \in \mathbb{R}$

(evaluated at g inverse of y)

(ii) g is differentiable

at every point $x = g^{-1}(y)$

& the derivative is not zero

Case 2: X continuous, Y discrete

$R_X \equiv X(S)$

then $R_Y = g(X(S))$

* find R_Y (the range space of Y)

where $R_Y = g(X(S))$, or some subset occurring w/ prob 1.

then $\forall y \in R_Y$ $\underbrace{P(Y = y)}_{\text{pmf of } Y} = P(g(X) = y) = P(X \in D_y)$,

where D_y is the set of all x in R s.t. $g(x) = y$

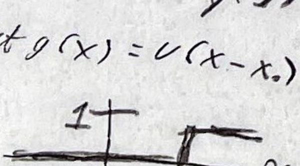
$D_y = \{x \in \mathbb{R} : g(x) = y\}$

then let $P_Y(y) = \int_{D_y} f_X(x)\,dx$, then $R_Y = \{0, 1\}$

let $g(x) = u(x - x_0)$

$D_0 = \{x \in \mathbb{R} \mid x < x_0\}$
$= (-\infty, x_0)$

$D_1 = \{x \in \mathbb{R} \mid x \ge x_0\}$
$= [x_0, \infty)$



$$\therefore P_Y(y) = \begin{cases} \int_{-\infty}^{x_0} f_X(x)\,dx, & y=0 \\ \int_{x_0}^{\infty} f_X(x)\,dx, & y=1 \end{cases}$$

(y) Fn of RV

Case 3: $X, Y \leftarrow$ discrete RVs

* find $R_Y \Rightarrow R_Y = g(X(S))$

$$\underset{\text{pmf of } y}{P(Y=y)} = P(g(X)=y) = P(\{X \in R : g(x)=y\}$$

$$= P(X \in D_y),\ D_y \subset R_x,\ \forall y \in R_y$$

$$\Rightarrow \boxed{P_Y(y) = \sum_{x \in D_y} P_X(x) = \sum_{x \in R_x : g(x)=y} P_X(x),\ \forall y \in R_y}$$

Example let X be the value of a die rolled

$\mathbb{I}[\ldots]$

let $Y = \begin{cases} 1 & \text{if } X \text{ is odd} \\ 0 & \text{if } X \text{ is even} \end{cases} \Rightarrow R_X = \{1,2,3,4,5,6\}$

$R_Y = \{0,1\}$

$\&\ g(x) = X \% 2$

$$P_Y(0) = \sum_{x \in R_X : x \bmod(2) = 0} P_X(x) = P_X(2) + P_X(4) + P_X(6)$$

y is zero if x is even

x mod(2)
(x mod 2) = {0

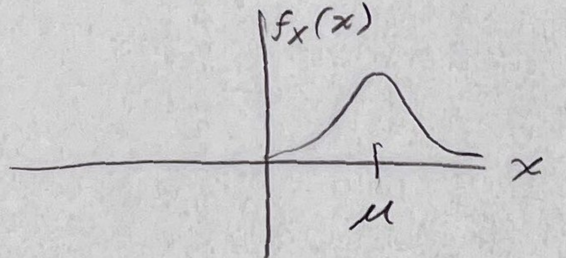
$$P_Y(1) = \sum_{x \in R_x} P_X(x) = P_X(1) + P_X(3) + P_X(5)$$

02/16

1) Gaussian (Normal) ← Continuous RV

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right], \forall x \in \mathbb{R}$$

density function



$\mu, \sigma \in \mathbb{R}$, $\sigma > 0$

(constant real #'s)

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(r-\mu)^2/2\sigma^2}\, dr$$

← this integral has no closed form solution

(integrate density function to get the probability) for gaussian distribution

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Let $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-r^2/2}\, dr$ ← [CDF of a gaussian RV w/ $\mu = 0$, & $\sigma = 1$]

→ lookup tables for Φ for some value of x

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if $\mu \neq 0$ or $\sigma \neq 1$

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$N(\mu, \sigma^2)$

normal gaussian distribution

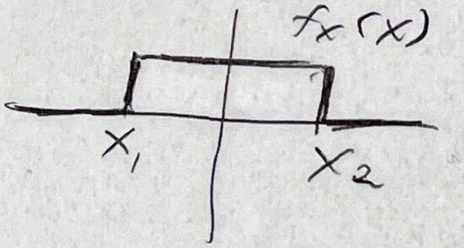
02/18

2) Uniform ← Continuous form:

(RV that are equally likely within an interval)

$$f_X(x) = \begin{cases} \dfrac{1}{x_2 - x_1}, & x_1 < X \le x_2 \\ 0, & \text{o.w.} \end{cases}$$

(for some x_1 & $x_2 \in \mathbb{R}$, $x_1 < x_2$)



~~cdf = 0 for x < x_1, 1 for x_1 <~~

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discrete form: $R_X = \{x_1, \ldots, x_n\}$

finite $n \ge 1$

pmf $p_X(x_i) = \frac{1}{n} \quad \forall x_i \in R_X$

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3) Binomial ← Discrete

used to model the # of successes within a Bernoulli Trial

$R_X = \{0, \ldots, n\}$

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$k = 0, \ldots, n$

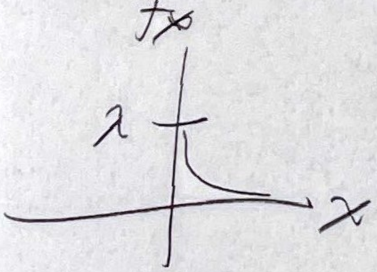
$p \in [0, 1]$, $n \ge 1$, finite

02/16

4) Exponential ← Continuous

used to model lifetime of devices/systems or times between an occurrence of an event

density function:



$$f_X(x) = \lambda e^{-\lambda x} \, u(x)$$

$$\lambda \in \mathbb{R}, \quad \lambda > 0$$

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5) Poisson ← Discrete

$$R_X = \{0, 1, 2, \dots\}$$

values that the RV can take with the probability 1

$$p_X(k) = \frac{e^{-\lambda} \lambda^k}{k!}, \quad \lambda > 0$$

$$k = 0, 1, \dots$$

used to model # of occurrences of a certain event in a given space/time interval

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6) Geometric ← Discrete

to model # of Bernoulli trials until the first success

Form 1: $R_X = \{0, 1, 2, \dots\}$

$$p_X(k) = p(1-p)^k$$

Form 2: $R_X = \{1, 2, \dots\}$

$$p_X(k) = p(1-p)^{k-1}$$

Expectation:

("Expected value of x")

for a continuous rv $\longrightarrow$ $\boxed{E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx}$ $\leftarrow$ (def for any rv)

for a discrete rv $\longrightarrow$ $E[X] = \sum_{x \in R_X} x\, p_X(x)$ $\leftarrow$ (if x is a discrete rv)

$E[X]$ or EX or $\bar{X}$ or m_X or μ_X

If rv X is an exponential w/ parameter λ:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{0}^{\infty} x \lambda e^{-\lambda x}\, dx = \ldots = \frac{1}{\lambda}$$

let $\lambda = \frac{1}{\mu}$

we can write the density fn $f_X(x) = \frac{1}{\mu} e^{-x/\mu}$

~~let $\mu = \frac{1}{\lambda}$~~

If rv X is discrete w/ $R_X = \{1, \ldots, n\}$ (where n is finite $\geq$ one)

$$E[X] = \sum_{k=1}^{n} (k)\left(\frac{1}{n}\right) = \frac{n+1}{2}$$

let $Y = g(X)$ (generalized)

g leads to Y being a valid RV

$$\Rightarrow E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$E[Y]$ or $E[g(x)]$ $= \sum_{x \in R_X} (g(x))\, p_X(x)$ if discrete rv